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The stability of an inverted pendulum with a vibrating suspension point☆

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Abstract

The problem of stabilizing the upper vertical (inverted) position of a pendulum using vibration of the suspension point is considered. The periodic function describing the vibrations of the suspension point is assumed to be arbitrary but possessing small amplitudes, and slight viscous damping is taken into account. A formula is obtained for the limit of the region of stability of the solutions of Hill's equation with damping in the neighbourhood of the zeroth natural frequency. The analytical and numerical results are compared and show good agreement. An asymptotic formula is derived for the critical stabilization frequency of the upper vertical position of the pendulum. It is shown that the effect of viscous damping on the critical frequency is of the third-order of smallness and, in all the examples considered, when viscous damping is taken into account the critical frequency increases.

The oscillations and stability of a pendulum with an oscillating suspension point have been investigated in many papers.[1–3] This problem is of interest at the present time in modern devices, namely, charged-particle traps in an electromagnetic field[14] and applications in quantum mechanics.[15,16] The main difference between the present paper and previous ones is that viscous damping is taken into account, the arbitrary nature of the periodic excitation function and the method of investigating the stability.

1. Basic relations

Plane oscillations of a physical pendulum about the upper vertical (unstable) equilibrium position with an oscillating suspension point are described by the equation

$$I\theta_{tt} + c\theta_t - mr(g + z_{tt})\sin\theta = 0. \tag{1.1}$$

Here I and m are the moment of inertia and the mass of the pendulum, θ is the angle of deflection of the pendulum from the upper vertical position, c is the coefficient of viscous damping, r is the distance from the suspension point to the centre of mass, g is the acceleration due to gravity, and z is the vertical displacement of the suspension point, which, by assumption, obeys the law

$$z = a\phi(\Omega t), \tag{1.2}$$

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where Ω is the excitation frequency and $\phi(\tau)$ is an arbitrary piecewise-continuous 2π-periodic function. The amplitude a and the coefficient of friction c are assumed to be small. For convenience we will introduce the following notation

$$\varphi = -\phi_{tt}. \tag{1.3}$$

The mean value of the 2π-periodic function $\varphi(\tau)$ is assumed to be equal to zero

$$\int_0^{2\pi} \varphi(\tau)d\tau = 0. \tag{1.4}$$

We will introduce the dimensionless time and parameters

$$\tau = \Omega t, \quad \beta = \frac{c}{I\Omega}, \quad \varepsilon = \frac{a\Omega_0^2}{g}, \quad \omega = \frac{\Omega_0}{\Omega}; \quad \Omega_0 = \sqrt{\frac{mrg}{I}}. \tag{1.5}$$

where Ω_0 is the natural frequency of free oscillations of the pendulum with a fixed suspension point about the lower vertical equilibrium position.

In the case of a mathematical pendulum

$$I = mr^2, \quad \Omega_0 = \sqrt{\frac{g}{r}}, \quad \beta = \frac{c}{mr^2\Omega}, \quad \varepsilon = \frac{a}{r},$$

where r is the length of the pendulum.

In the notation of (1.2), (1.3) and (1.5), Eq. (1.1) takes the form

$$\ddot{\theta} + \beta\dot{\theta} - [\omega^2 - \varepsilon\varphi(\tau)]\sin\theta = 0, \tag{1.6}$$

where the dot denotes differentiation with respect to τ. The coefficients of this equation depend explicitly on the 2π-periodic function $\varphi(\tau)$ and the three dimensionless parameters ε, β and ω, where ε and β are small compared with unity.

2. The stability region for Hill's equation with damping

According to Lyapunov's theorem of stability by the first approximation, the stability of the trivial solution $\theta = 0$ of non-linear Eq. (1.6) with periodic coefficients is determined by the stability of the solution of the linearized equation

$$\ddot{\theta} + \beta\dot{\theta} + [\mu + \varepsilon\varphi(\tau)]\theta = 0; \quad \mu = -\omega^2. \tag{2.1}$$

This is Hill's equation with damping.[7,9,17]

When $\varepsilon = 0$ and $\beta = 0$ the upper vertical equilibrium position of the pendulum $\theta = 0$ is unstable. We will investigate the possibility of stabilizing the pendulum using high-frequency oscillations of the suspension point. This corresponds to the parameters of the problem being close to the point

$$\varepsilon = 0, \quad \beta = 0, \quad \mu = 0. \tag{2.2}$$

To find the region of stability in the neighbourhood of this point we will use a method based on calculating the derivatives of the monodromy matrix and analysing the behaviour of the multipliers.[17–19]

We will introduce the variables $x_1 = \theta$, $x_2 = \dot{\theta}$ and write Eq. (2.1) in the form of a first-order system of equations

$$\dot{\mathbf{x}} = \mathbf{G}\mathbf{x}; \quad \mathbf{x} = \begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix}, \quad \mathbf{G} = \begin{Vmatrix} 0 & 1 \\ -\mu - \varepsilon\varphi(\tau) & -\beta \end{Vmatrix}. \tag{2.3}$$

The fundamental matrix $\mathbf{X}(\tau)$ (the matricant) of system (2.3) is found from the matrix differential equation with initial condition

$$\dot{\mathbf{X}} = \mathbf{G}\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I}, \tag{2.4}$$

where $\mathbf{I}$ is the identity matrix. The monodromy (Floquet) matrix is defined by the equation $\mathbf{F}=\mathbf{X}(T)$. To investigate the stability of linear system (2.3) we will use Floquet's theory,[20] according to which a linear system with periodic coefficients is stable if all the eigenvalues (multipliers) of the monodromy matrix $\mathbf{F}$ are less than unity in modulus, and unstable if at least one multiplier is greater than unity in modulus.

Suppose we know the monodromy matrix $\mathbf{F}_0=\mathbf{F}(\mathbf{p}_0)$ for a certain n-dimensional vector of the real parameters $\mathbf{p}_0$. We will give the vector of the parameters an increment in the form $\mathbf{p}=\mathbf{p}_0+\Delta\mathbf{p}$, as a result of which the matrix $\mathbf{G}$ and, consequently, the matricant $\mathbf{X}(\tau)$ obtain increments. This correspondingly leads to a change in the monodromy matrix $\mathbf{F}$. Expressions for the first and second derivatives of the monodromy matrix with respect to the parameters were obtained in the form of integrals over a period[18]

$$\mathbf{F}_{,k} = \mathbf{F}_0\mathbf{P}_k \tag{2.5}$$

$$\mathbf{F}_{,ij} = \mathbf{F}_0(\mathbf{K}_{ij}+\hat{\mathbf{K}}_{ij}+\hat{\mathbf{K}}_{ji}), \tag{2.6}$$

where

$$\mathbf{F}_{,k} = \frac{\partial\mathbf{F}}{\partial p_k}, \quad \mathbf{F}_{,ij} = \frac{\partial^2\mathbf{F}}{\partial p_i\partial p_j}, \quad \mathbf{P}_k = \int_0^T \mathbf{H}_k(\tau)d\tau, \quad \mathbf{K}_{ij} = \int_0^T \mathbf{H}_{ij}(\tau)d\tau$$

$$\hat{\mathbf{K}}_{ij} = \int_0^T \mathbf{H}_i(\tau)\left(\int_0^\tau \mathbf{H}_j(\zeta)d\zeta\right)d\tau$$

$$\mathbf{H}_k(\tau) = \mathbf{X}_0^{-1}(\tau)\mathbf{G}_{,k}(\mathbf{p}_0,\tau)\mathbf{X}_0(\tau), \quad \mathbf{H}_{ij}(\tau) = \mathbf{X}_0^{-1}(\tau)\mathbf{G}_{,ij}(\mathbf{p}_0,\tau)\mathbf{X}_0(\tau), \quad i,j,k = 1,\ldots,n.$$

The zero subscript denotes that the corresponding quantity is taken when $\mathbf{p}=\mathbf{p}_0$. The formula for the n-th order derivatives contains n-tuple integrals.[19]

Note that, to obtain the derivatives of the monodromy matrix it is only necessary to know the matricant $\mathbf{X}_0(\tau)$ and the derivatives of the matrix $\mathbf{G}$ with respect to the parameters, taken at $\mathbf{p}=\mathbf{p}_0$. Expanding the monodromy matrix in the neighbourhood of the point $\mathbf{p}_0$

$$\mathbf{F}(\mathbf{p}) = \mathbf{F}_0 + \sum_{k=1}^n \mathbf{F}_{,k}\Delta p_k + \frac{1}{2}\sum_{i,j=1}^n \mathbf{F}_{,ij}\Delta p_i\Delta p_j + \ldots, \tag{2.7}$$

we can estimate the behaviour of the multipliers (the eigenvalues of the monodromy matrix $\mathbf{F}$), responsible for the stability of system (2.3) when the parameters change.

From Eqs. (2.3) and (2.4) at the point (2.2) we obtain the matricant and the matrix inverse to it

$$\mathbf{X}_0(\tau) = \begin{Vmatrix} 1 & \tau \\ 0 & 1 \end{Vmatrix}, \quad \mathbf{X}_0^{-1}(\tau) = \begin{Vmatrix} 1 & -\tau \\ 0 & 1 \end{Vmatrix}. \tag{2.8}$$

Hence, the monodromy matrix at the point (2.2) is equal to

$$\mathbf{F}_0 = \mathbf{X}_0(2\pi). \tag{2.9}$$

The eigenvalues of this matrix are double multipliers $\rho_1=\rho_2=1$ with a single eigenvector (strong interaction[18]), and hence for a small change in the parameters ε, β and μ in the neighbourhood of the origin of coordinates $\mathbf{p}_0=(0,0,0)$ of parameter space, the multipliers either diverge along the real axis of the complex plane on different sides (instability), or become complex conjugate quantities and diverge in a direction perpendicular to the real axis. Stability means that in this case the multipliers remain inside the unit circle.

To find the stability region we expand the monodromy matrix $\mathbf{F}$ in the neighbourhood of the point $\mathbf{p}_0$ in a Taylor series (2.7) with respect to the parameters ε, β and μ. Then, by relations (2.7) and (2.9), we represent the monodromy matrix in the form

$$\mathbf{F}(\mathbf{p}) = \begin{Vmatrix} 1+\alpha_{11} & 2\pi+\alpha_{12} \\ \alpha_{21} & 1+\alpha_{22} \end{Vmatrix}, \tag{2.10}$$

where α_{ij} are small quantities. By Liouville's theorem[20]

$$\det\mathbf{F}(\mathbf{p}) = \exp\left(\int_0^{2\pi} \mathrm{tr}\mathbf{G}dt\right) = \exp(-2\pi\beta);$$

and hence the solution of the characteristic equation $\det||\mathbf{F}(\mathbf{p}) - \rho\mathbf{I}|| = 0$ can be written in the form

$$\rho = \mathrm{tr}\mathbf{F}/2 \pm \sqrt{(\mathrm{tr}\mathbf{F}/2)^2 - \exp(-2\pi\beta)}. \tag{2.11}$$

The condition for asymptotic stability $|\rho| < 1$ can be written in the form of a system of inequalities, connecting the diagonal elements of the monodromy matrix.

For a negative discriminant we have

$$\alpha_{11} + \alpha_{22} < 0, \quad \alpha_{11} + \alpha_{22} < 2(\exp(-\pi\beta) - 1), \quad \exp(-2\pi\beta) < 1. \tag{2.12}$$

Note that the last condition of (2.12), imposed on the modulus of the multipliers, is satisfied for any damping factor $\beta > 0$. Hence it follows that the system is asymptotically stable for a negative discriminant.

For a non-negative discriminant the stability conditions have the form

$$\alpha_{11} + \alpha_{22} < 0, \quad \alpha_{11} + \alpha_{22} \geq 2(\exp(-\pi\beta) - 1), \quad \exp(-2\pi\beta) - 1 > \alpha_{11} + \alpha_{22}. \tag{2.13}$$

It is the third condition of (2.13) that defines the limit of the stability region.

From formulae (2.5) and (2.6) and also the relations for higher-order derivatives,[19] and using expressions (2.3), (2.8) and (2.9) we can calculate the values of the derivatives of the monodromy matrix with respect to the parameters up to the fourth order for $\mathbf{p} = \mathbf{p}_0$ and hence obtain α_{ii}. As a result the condition governing stability (the third condition of (2.13)) takes the form

$$\begin{aligned} &4\pi^4\mu^2/3 - 4\pi^2\mu(1 - \pi\beta + 2\pi^2\beta^2/3) + a\varepsilon^2\mu + \\ &+ A\varepsilon^2 + B\varepsilon^2\beta^2 + M\varepsilon^3 + C\varepsilon^4 + D\varepsilon^2\beta + S\varepsilon^3\beta < 0, \end{aligned} \tag{2.14}$$

where

$$A = \left(\int_0^{2\pi} t\varphi(t)dt\right)^2 - 4\pi\int_0^{2\pi}\varphi(t)\int_0^t \tau\varphi(\tau)d\tau dt.$$

The other coefficients of the polynomial on the left-hand side of inequality (2.14) can be expressed in terms of the trace of the derivatives of the monodromy matrix with respect to the parameters, but have a quite complex form and will not be given here. Using the conversion of multiple integrals, it can be shown by means of integration by parts that the coefficients A and D and M and S are connected by the relations

$$\pi A + D = 0, \quad \pi M + S = 0. \tag{2.15}$$

It then follows from condition (2.14) that the stability region is given by the inequality

$$\mu > F\varepsilon^2 + N\varepsilon^3 + K\varepsilon^2\beta^2 + L\varepsilon^4. \tag{2.16}$$

The coefficients of the polynomial on the right-hand side of this inequality are found by substituting (2.16) into the equation of the boundary, which is obtained from (2.14) by changing the inequality sign into an equality and equating corresponding terms of the same order of smallness to zero. We finally obtain

$$F = \frac{A}{4\pi^2}, \quad N = \frac{M}{4\pi^2}, \quad K = \frac{B}{4\pi^2} - \frac{2\pi^2 F}{3}, \quad L = \frac{\pi^2 F^2}{3} + \frac{aF + C}{4\pi^2}. \tag{2.17}$$

Hence, it follows from condition (2.16) for an arbitrary periodic function $\varphi(\tau)$ in the neighbourhood of the point $\mathbf{p}_0 = (0,0,0)$ that, apart from quantities of the third order of smallness, the stability region is given by the inequality

$$\mu > F\varepsilon^2; \quad F = \left(\frac{1}{2\pi}\int_0^{2\pi} t\varphi(t)dt\right)^2 - \frac{1}{\pi}\int_0^{2\pi}\varphi(t)\int_0^t \tau\varphi(\tau)d\tau dt. \tag{2.18}$$

Note that, by our assumption, condition (1.4) is satisfied for the function $\varphi(\tau)$. It can be shown that, for periodic functions, parts of which may be reflected symmetrically in the time axis and superimposed, the constant $F < 0$.

For a periodic function $\varphi(\tau) = \cos\tau$ Eq. (2.1) is called the Mathieu-Hill equation. The stability condition (the last inequality of (2.13)) for this equation leads to the relation

$$\frac{\pi^2\mu^2}{3} - \mu(1-\gamma_1) - \frac{\varepsilon^2}{2} + \gamma_2 < 0, \tag{2.19}$$

where we have introduced the following notation

$$\gamma_1 = \pi\beta - \frac{2\pi^2\beta^2}{3} + \left(\frac{\pi^2}{3} - 2\right)\varepsilon^2, \quad \gamma_2 = \frac{\pi\varepsilon^2\beta}{2} + \left(\frac{1}{2} - \frac{\pi^2}{3}\right)\varepsilon^2\beta^2 + \left(\frac{\pi^2}{3} - \frac{25}{8}\right)\frac{\varepsilon^4}{4}.$$

Correspondingly, by relations (2.16) and (2.17), we obtain the stability condition in the form

$$\mu > -\frac{\varepsilon^2}{2} + \frac{\varepsilon^2\beta^2}{2} + \frac{7\varepsilon^4}{32}. \tag{2.20}$$

Hence it follows that when damping is taken into account the region of stability in the neighbourhood of the point $\mathbf{p}_0 = (0,0,0)$ contracts when $\mu < 0$ and damping is a destabilizing factor, unlike the case when $\mu > 0$.[17] Note that the assertion that damping has no effect on the boundary of the stability region for the Mathieu–Hill equation for negative values of μ,[7] according to the above results, is only true to a first approximation.

In Fig. 1 we show the effect of damping on the boundary of the stability region for a periodic function $\varphi(\tau) = \cos\tau$. The boundary of the stability region, obtained using analytical formula (2.20), is represented by the thick curve, while the instability regions, obtained using numerical integration of Eqs. (2.2) and (2.3) and calculations of the monodromy matrix $\mathbf{F}$ and of the absolute values of the multipliers, are shown hatched. It follows from Fig. 1 that the boundaries of the stability region, obtained analytically and numerically, agree well with one another up to values of $\varepsilon \approx 0.5$, $\beta \approx 0.4$ and $\mu \approx -0.15$. Note that when $\beta = 0$ we have part of the Ince-Strutt diagram,[7,10] which, taking into account the sixth-order term in ε, is described by the asymptotic formula

$$\mu > -\frac{\varepsilon^2}{2} + \frac{7\varepsilon^4}{32} - \frac{29\varepsilon^6}{144}. \tag{2.21}$$

In the example considered above the cubic term in ε in formula (2.16) is not present ($N = 0$). It is easy to prove that the absence of odd terms in formula (2.16) occurs for arbitrary 2π-periodic functions with symmetry $\varphi(\tau + \pi) = -\varphi(\tau)$. In fact, making the time conversion $t = \tau + \pi$ in Eq. (2.1) and using the symmetry property, we arrive at the equivalent equation $\ddot{\theta} + \beta\dot{\theta} + [\mu - \varepsilon\varphi(t)]\theta = 0$. Hence it follows that the stability region is symmetrical about the μ axis, and hence there are no odd terms in ε in formula (2.16).

For a periodic function with zero mean value; not having this symmetry,

$$\varphi(\tau) = \frac{1}{4}\left(\frac{\tau}{\pi}\right)^3 - \frac{1}{2}, \quad 0 \le \tau \le 2\pi \tag{2.22}$$

and the stability region, using relations (2.16) and (2.17), is determined by the condition

$$\mu > -\frac{19\pi^2\varepsilon^2}{900} - \frac{41\pi^4\varepsilon^3}{103950} + \frac{139\pi^4\varepsilon^2\beta^2}{69300} + \frac{1074071\pi^6\varepsilon^4}{40439148750}. \tag{2.23}$$

As in the previous case, when damping is taken into account the stability region in the neighbourhood of the point $\mathbf{p}_0 = (0,0,0)$ contracts, and damping is a destabilizing factor. In Fig. 2 the boundary of the stability region, obtained using

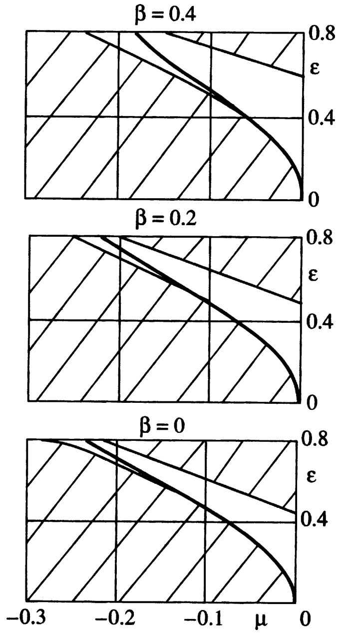

Fig. 1.

analytical formula (2.23), is indicated by the heavy curve, while the stability regions, obtained numerical integration, are shown hatched. There is a small asymmetry of the stability region about the μ axis. A comparison of the analytical and numerical results for the boundary of the stability region confirms their excellent agreement up to values of $\varepsilon \approx 0.8$, $\beta \approx 0.4$ and $\mu \approx = -0.1$.

The results obtained supplement the investigation of the stability and instability regions for Hill's equation with damping[17] for the case of zero frequency $\omega = 0$.

Remark. If the mean value of the periodic function $\varphi(\tau)$ is not equal to zero, then, using the replacement of variables

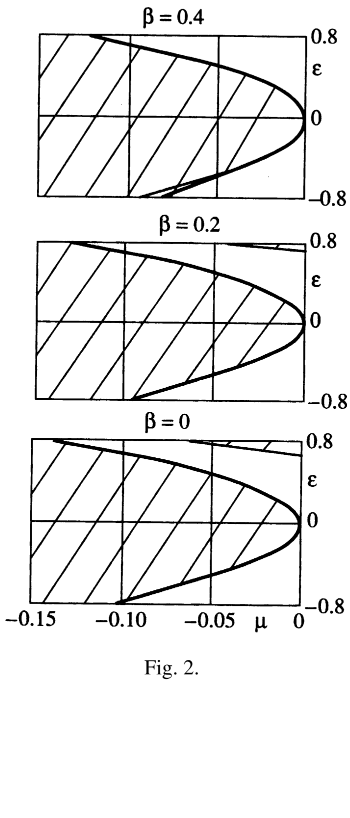

Fig. 2.

$$\bar{\mu} = \mu + \varepsilon\varphi_0, \quad \tilde{\varphi}(\tau) = \varphi(\tau) - \varphi_0, \quad \varphi_0 = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\tau)d\tau$$

Hill's equation can be reduced to the form (2.1) with periodic function $\tilde{\varphi}(\tau)$, having zero mean value. In this case the equation of the boundary of the stability region in μ, ε variables will also contain a linear term in ε.

3. Stabilization of the pendulum

Taking relations (1.4) into account, for convenience we can write the factor β in the form

$$\beta = \beta_0\omega, \quad \beta_0 = c/(I\Omega_0). \tag{3.1}$$

Now the damping factor β_0 is independent of the excitation frequency Ω. Then, taking into account the fact that $\mu = -\omega^2$, and calculating the derivatives of the monodromy matrix with respect to ε of the fifth and sixth order, using formula (2.16) we obtain the inequality

$$-\omega^2(1 + K\varepsilon^2\beta_0^2) > \varepsilon^2(F + N\varepsilon + L\varepsilon^2 + V\varepsilon^3 + H\varepsilon^4). \tag{3.2}$$

The coefficients F, N, K and L are given by formulae (2.17), while the coefficients V and H are calculated similarly. Extracting the root of both sides of inequality (3.2) and expanding the radicals in series, taking relations (1.4) into account we obtain the stabilization range of the pendulum, apart from terms of the fourth order of smallness, if the quantities ε and β_0 are assumed to be of the same order. It is given by the condition

$$\begin{aligned}\frac{\Omega}{\Omega_0} > T\bigg[&\frac{2F}{\varepsilon} - N + \varepsilon\left(\frac{3N^2}{4F} - L\right) + \varepsilon^2\left(\frac{3NL}{2F} - \frac{5N^3}{8F^2} - V\right) + \varepsilon\beta_0^2KF + \\ &+ \varepsilon^3\left(\frac{3(L^2 + 2NV)}{4F} + \frac{35N^4}{64F^3} - \frac{15N^2L}{8F^2} - H\right)\bigg]; \quad T = \frac{1}{2F\sqrt{-F}}.\end{aligned} \tag{3.3}$$

Hence it follows that, at a sufficiently high frequency of vibration of the suspension point, the upper vertical position of the pendulum becomes stable (it is assumed that $F < 0$). When $K > 0$ slight damping increases the critical stabilization frequency. However, as formula (3.3) shows, the effect of damping on the critical stabilization frequency is very small (of the third order of smallness).

Using relation (3.3) we obtain for the function $\varphi(\tau) = \cos\tau$

$$\frac{\Omega}{\Omega_0} > \sqrt{2}\left[\frac{1}{\varepsilon} + \frac{7\varepsilon}{32} + \frac{\varepsilon\beta_0^2}{4} - \frac{2389\varepsilon^3}{18432}\right]. \tag{3.4}$$

Hence, at a fairly high vibration frequency of the suspension point the upper vertical position of the pendulum becomes stable and slight damping increases the critical stabilization frequency. The term of the order of ε^{-1} on the right-hand side of equality (3.4) is identical with that obtained previously,[1–11] and the next two terms are also known.[8] Owing to the above-mentioned symmetry the coefficients $N = V = 0$, and hence in formula (3.4) there are no terms of the zeroth and second order in ε.

Considering a mathematical pendulum with a piecewise-constant excitation function

$$\varphi(\tau) = \begin{cases} 1, & 0 < \tau \le \pi \\ -1, & \pi < \tau < 2\pi, \end{cases} \tag{3.5}$$

for which $F = -\pi^2/12$, using expression (3.3) we obtain the stabilization condition in the form

$$\frac{\Omega}{\Omega_0} > \sqrt{3}\left[\frac{2}{\pi\varepsilon} + \frac{\pi^3\varepsilon}{126} + \frac{\pi^3\varepsilon\beta_0^2}{120} - \frac{79\pi^7\varepsilon^3}{970200}\right]. \tag{3.6}$$

In the first approximation the critical stabilization frequency for this case is identical with the well-known result in Ref. 12 and is $\sqrt{2}$ times less than the critical frequency derived in Ref. 13.

For periodic function (2.22) not possessing symmetry, using the calculations described above we obtain the stabilization condition in the form

$$\frac{\Omega}{\Omega_0} > \frac{2.19}{\varepsilon} - 0.202 + 0.162\varepsilon + 0.045\varepsilon^2 + 0.214\varepsilon\beta_0^2 - 0.028\varepsilon^3. \tag{3.7}$$

From a comparison with the previous examples we note the presence of terms of the zeroth and second powers in ε. As previously, taking damping into account leads to an increase in the critical stabilization frequency.

Note that in practical cases the parameters of the amplitude and damping satisfy the inequalities $\varepsilon < 0.1$ and $\beta_0 < 0.1$, and hence the asymptotic formulae obtained give a practical solution of the problem of stabilizing an inverted pendulum using excitation of oscillations of the suspension point with high accuracy.

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